Math 250 2.5 Limits at Infinity

Objectives

1) Find limits at infinity (as x goes to either ∞ or $-\infty$), also called "end behavior"
   a. Finite limits at infinity: $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ means horizontal asymptote $y = L$
      i. $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$ provided $n > 0$
   b. 'Infinite limits at infinity: $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$ or $\lim_{x\to-\infty} f(x) = \infty$ or $\lim_{x\to-\infty} f(x) = -\infty$
      i. $\lim_{x\to\pm\infty} x^n = \infty$ if n is even
      ii. $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ if n is odd
      iii. $\lim_{x\to\pm\infty} p(x)$ for a polynomial depends on the degree n and the leading coefficient a_n

	$a_n > 0$	$a_n < 0$
n odd	$\lim_{x\to\infty} p(x) = +\infty$, $\lim_{x\to-\infty} p(x) = -\infty$	$\lim_{x\to\infty} p(x) = -\infty$ $\lim_{x\to-\infty} p(x) = +\infty$
n even	$\lim_{x\to\infty} p(x) = +\infty$ $\lim_{x\to-\infty} p(x) = +\infty$	$\lim_{x\to\infty} p(x) = -\infty$ $\lim_{x\to-\infty} p(x) = -\infty$

1.

   c. Oscillating limits that do not exist (DNE) at infinity
2) Analytic methods for limits at infinity
   a. Polynomial functions
   b. Rational functions
   c. Fractional functions containing square roots
3) Determine equations of horizontal asymptotes or slant (oblique) asymptotes

Examples & Practice:

1) $f(x) = \dfrac{5x-4}{x+3}$
   a. Graph the function in GC.
   b. Approximate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ from GC
   c. Does this function have a horizontal or oblique asymptote or neither?
   d. If applicable, write the equation of the end-behavior asymptote(s).
   e. Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ analytically.

2) $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$
   a. Use analytic methods to find $\lim_{x\to\infty} f(x)$
   b. Use analytic methods to find $\lim_{x\to-\infty} f(x)$
   c. Does this function have a horizontal or oblique asymptote or neither?
   d. If applicable, write the equation of the end-behavior asymptote(s).

3) $f(x) = \dfrac{\sin x}{\sqrt{x}}$

a. Graph the function in GC.
b. Approximate $\lim_{x \to \infty} f(x)$ from GC
c. Use the Squeeze theorem to find $\lim_{x \to \infty} f(x)$
d. Does this function have a horizontal or oblique asymptote or neither?
e. If applicable, write the equation of the end-behavior asymptote(s).
f. Why did we not find $\lim_{x \to -\infty} f(x)$?

4) $f(x) = 3x^4 - 6x^2 + x - 10$

a. Find $\lim_{x \to \infty} f(x)$
b. Find $\lim_{x \to -\infty} f(x)$
c. Graph the function in GC to confirm answers.

5) $f(x) = -3x^5 - 6x^2 + x - 10$

a. Find $\lim_{x \to \infty} f(x)$
b. Find $\lim_{x \to -\infty} f(x)$
c. Graph the function in GC to confirm answers.

6) $f(x) = \dfrac{3x - 4}{x^2 + 3}$

a. Find $\lim_{x \to \infty} f(x)$
b. Find $\lim_{x \to -\infty} f(x)$
c. Does this function have a horizontal or oblique asymptote or neither?
d. If applicable, write the equation of the end-behavior asymptote(s).

7) $f(x) = \dfrac{3x^3 - 4}{2x + 3}$

a. Find $\lim_{x \to \infty} f(x)$
b. Find $\lim_{x \to -\infty} f(x)$
c. Does this function have a horizontal or oblique asymptote or neither?
d. If applicable, write the equation of the end-behavior asymptote(s).

8) $f(x) = \sin x$

a. Find $\lim_{x \to \infty} f(x)$
b. Find $\lim_{x \to -\infty} f(x)$
c. Does this function have a horizontal or oblique asymptote or neither?
d. If applicable, write the equation of the end-behavior asymptote(s).

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Recall from Precalculus:

End behavior of a rational function. polynomial / polynomial. (no $\sqrt{\ }$!)

- same horizontal (or slant) asymptote to the left as to the right.
  $x \to -\infty$ $x \to +\infty$

horizontal asymptote $y=0$ finite end behavior

- If degree of numerator is less than degree of denominator

  e.g. $f(x) = \dfrac{x^2+1}{3x^4-2x^3}$

  The function (y-coords) go to 0 as $x \to \infty$ or $x \to -\infty$. $\lim_{x\to\infty} f(x) = 0$

horizontal asymptote $y = \frac{3}{5}$ finite end behavior

- If the degree of the numerator is equal to the degree of the denominator,

  e.g. $g(x) = \dfrac{3x^4+2x^2}{5x^4-x^3+2}$

  The function y-coords go to a y-value that you get by reducing the ratio of leading terms:

  $\dfrac{3x^4}{5x^4} = \dfrac{3}{5}$ $y = \frac{3}{5}$ $\lim_{x\to\infty} g(x) = \frac{3}{5}$

  y-coords go to $\frac{3}{5}$ as $x \to +\infty$ or $x \to -\infty$.

infinite end behavior $\Rightarrow$ slant or oblique asymptote

- If the degree of the numerator is greater than the degree of the denominator

  e.g. $h(x) = \dfrac{5x^4-x^3}{3x^3+2x^2+1}$

  The y-coords are unbounded as $x \to \pm\infty$.

  $\lim_{x\to\infty} h(x) = \infty$ or $-\infty$

  This case gives an <u>infinite limit</u> <u>at infinity</u>.

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Example ① $f(x) = \dfrac{5x-4}{x+3}$

a) Identify vertical asymptote. Write its equation.
b) Identify horizontal asymptote. Write its equation.
c) Sketch graph.

a) vertical asymptote.

Step 1: check that no factors cancel (holes, not asymptotes).
Step 2: set denominator $=0$ and solve for variable.

$x+3=0$

$\boxed{x=-3}$ ← equation of vertical line which graph approaches.

General: $\boxed{x+3=0}$

b) horizontal asymptote:

Step 1: Determine degree of numerator and degree of denominator.

$\deg(5x-4) = 1$
$\deg(x+3) = 1.$

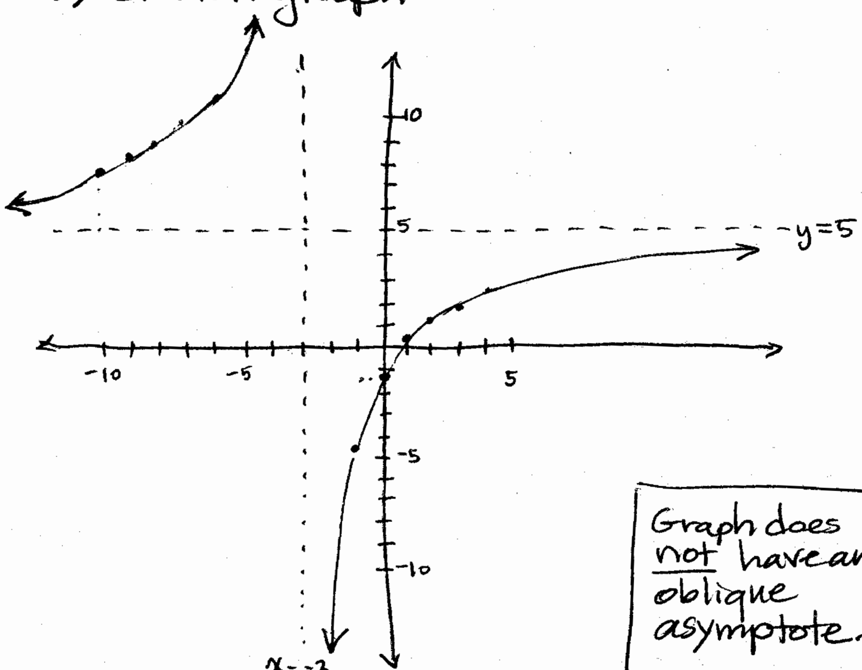
Step 2: Because degrees in step 1 are equal, write fraction of leading terms and reduce.

$\dfrac{5x}{x} = 5$

$\boxed{y=5}$ ← equation of horizontal line which graph approaches.

General: $\boxed{y-5=0}$ = end-behavior asymptote

c) sketch graph



x	approx f(x)
−10	7.7
−9	8.2
−8	8.8
−7	9.8
−6	11.3
−5	14.5
−4	24
−3	undef
−2	−14
−1	−4.5
0	−1.3
1	.25
2	1.2
3	1.8
4	2.3

Graph does not have an oblique asymptote.

Use graph to approximate these limits.

Review

$$\lim_{x \to -3^+} \frac{5x-4}{x+3} = \boxed{\begin{array}{l}-\infty \\ \text{DNE} \\ \text{unbounded}\end{array}}$$

x values move toward
x = -3 from right
graph goes down off grid

$$\lim_{x \to -3^-} \frac{5x-4}{x+3} = \boxed{\begin{array}{l}+\infty \\ \text{DNE} \\ \text{unbounded}\end{array}}$$

x-values move toward
x = -3 from left
graph goes up off grid

$$\lim_{x \to -3} \frac{5x-4}{x+3} = \boxed{\text{DNE, } L \neq R\text{, unbounded}}$$

2-sided limit
must approach
same value from
left as from right

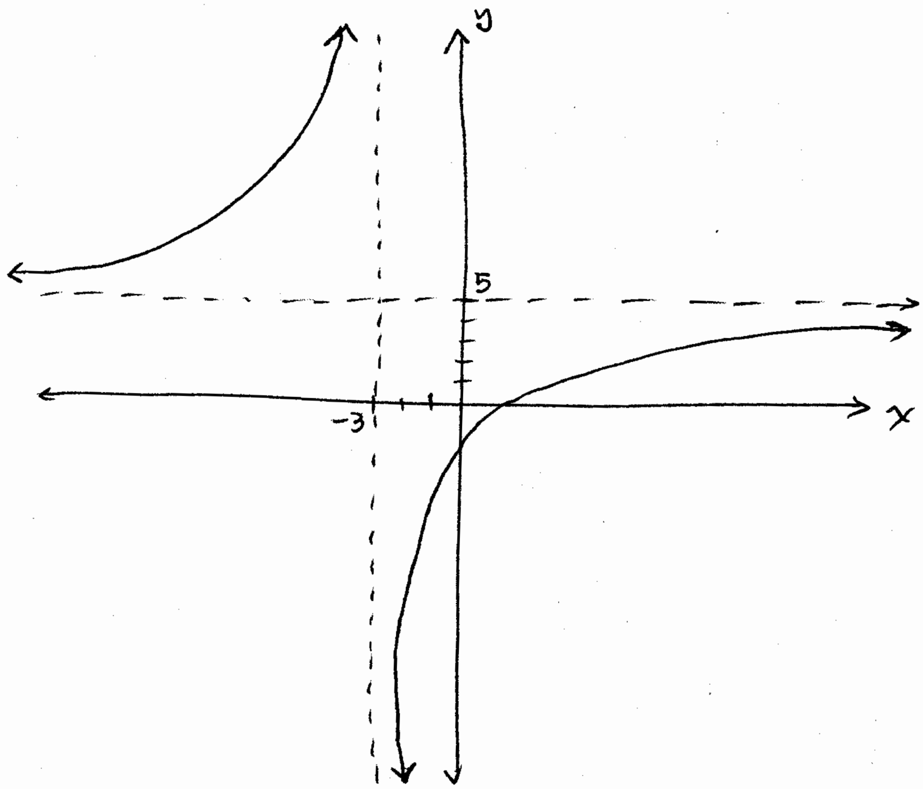
New

$$\lim_{x \to \infty} \frac{5x-4}{x+3} = \boxed{5}$$

x-values move right off
edge of grid, y-values
approach asymptote value

$$\lim_{x \to -\infty} \frac{5x-4}{x+3} = \boxed{5}$$

x-values move left off
edge of grid, y-values
approach asymptote value



Evaluate $f(x) = \frac{1}{x}$ to complete the tables and approximate the $\lim_{x \to \infty} \frac{1}{x}$, and $\lim_{x \to -\infty} \frac{1}{x}$.

x	f(x)
10	.1
100	.01
1000	.001
10,000	.0001
100,000	.00001
1,000,000	.000001
10,000,000	.0000001

x	f(x)
−10	−.1
−100	−.01
−1000	−.001
−10,000	−.0001
−100,000	−.00001
−1,000,000	−.000001
−10,000,000	−.0000001

$$\boxed{\begin{aligned} \lim_{x \to \infty} \frac{1}{x} &= 0 \\ \lim_{x \to -\infty} \frac{1}{x} &= 0 \end{aligned}}$$

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① Find limit analytically.

$$\lim_{x\to\infty} \frac{5x-4}{x+3}$$

step 1: Determine degree of denominator and find x^n where n is degree.

denom $x+3$ is degree 1

$x^n = x^1 = x.$

step 2: Divide every term of both numerator and denominator by x^n and simplify.

$$= \lim_{x\to\infty} \frac{\frac{5x}{x} - \frac{4}{x}}{\frac{x}{x} + \frac{3}{x}}$$

$$= \lim_{x\to\infty} \frac{5 - \frac{4}{x}}{1 + \frac{3}{x}}$$

step 3: Rewrite by separating, using properties of limits.

$$= \frac{\lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{4}{x}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{3}{x}}$$

step 4: Evaluate limits and do arithmetic

$$= \frac{5-0}{1+0}$$

$$= \boxed{5}$$

matches graph ☺

② $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$ ← highest degree term of denom, with $\sqrt{\ }$ is $\sqrt{x^6}$

Recall:

$$\sqrt{x^2} = |x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

So

$$\sqrt{x^4} = |x^2| = x^2$$

$$\sqrt{x^6} = |x^3| = \begin{cases} x^3 & x > 0 \\ -x^3 & x < 0 \end{cases}$$

$$\sqrt{x^8} = |x^4| = x^4$$

etc.

a) Divide numerator and denominator by power function determined by leading term of denominator.

$$\lim_{x \to \infty} \frac{\dfrac{10x^3 - 3x^2 + 8}{\sqrt{x^6}}}{\dfrac{\sqrt{25x^6 + x^4 + 2}}{\sqrt{x^6}}}$$

since $x \to \infty$ we have positive $x > 0$. $\sqrt{x^6} = x^3$ in this limit's numerator.

$$= \lim_{x \to \infty} \frac{\dfrac{10x^3}{x^3} - \dfrac{3x^2}{x^3} + \dfrac{8}{x^3}}{\sqrt{\dfrac{25x^6}{x^6} + \dfrac{x^4}{x^6} + \dfrac{2}{x^6}}}$$

$\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ Quotient property of radicals

$$= \lim_{x \to \infty} \frac{10 - \dfrac{3}{x} + \dfrac{8}{x^3}}{\sqrt{25 + \dfrac{1}{x^2} + \dfrac{2}{x^6}}}$$

simplify each fraction.

$$= \frac{\lim\limits_{x \to \infty} 10 - 3 \cdot \left[\lim\limits_{x \to \infty} \dfrac{1}{x}\right] + 8 \cdot \left[\lim\limits_{x \to \infty} \dfrac{1}{x^3}\right]}{\sqrt{\lim\limits_{x \to \infty} 25 + \lim\limits_{x \to \infty} \dfrac{1}{x^2} + 2 \cdot \left[\lim\limits_{x \to \infty} \dfrac{1}{x^6}\right]}}$$

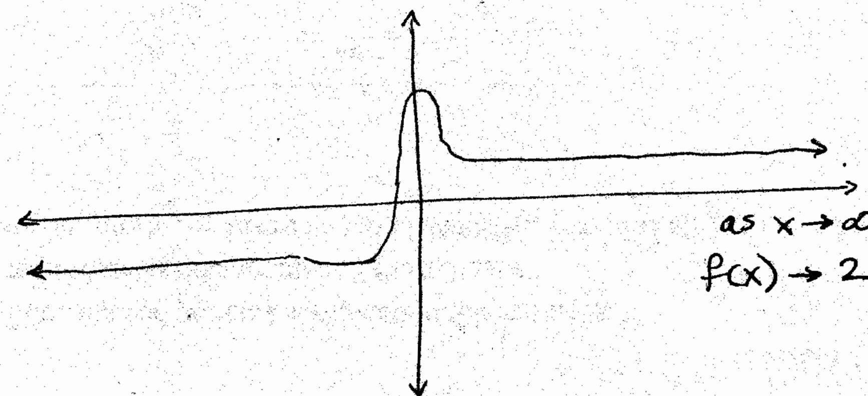
$$= \frac{10 - 3\cdot 0 + 8\cdot 0}{\sqrt{25 + 0 + 2\cdot 0}}$$

Confirm on GC

$$= \frac{10}{\sqrt{25}}$$

$$= \frac{10}{5}$$

$$= \boxed{2}$$



as $x \to \infty$
$f(x) \to 2$

b) $$\lim_{x\to -\infty} \frac{\dfrac{10x^3 - 3x^2 + 8}{\sqrt{x^6}}}{\dfrac{\sqrt{25x^6 + x^4 + 2}}{\sqrt{x^6}}}$$

As $x \to -\infty$
$x < 0$ is negative
so
$\sqrt{x^6} = |x^3| = -x^3$
in numerator

$$= \lim_{x\to -\infty} \frac{\dfrac{10x^3}{-x^3} - \dfrac{3x^2}{-x^3} + \dfrac{8}{-x^3}}{\sqrt{\dfrac{25x^6}{x^6} + \dfrac{x^4}{x^6} + \dfrac{2}{x^6}}}$$

$$= \lim_{x\to -\infty} \frac{-10 + \dfrac{3}{x} - \dfrac{8}{x^3}}{\sqrt{25 + \dfrac{1}{x^2} + \dfrac{2}{x^6}}}$$

simplify

$$= \frac{\lim_{x\to-\infty} -10 + 3\cdot\left[\lim_{x\to-\infty} \frac{1}{x}\right] - 8\left[\lim_{x\to-\infty}\frac{1}{x^3}\right]}{\sqrt{\lim_{x\to-\infty} 25 + \left[\lim_{x\to-\infty}\frac{1}{x^2}\right] + 2\cdot\left[\lim_{x\to-\infty}\frac{1}{x^6}\right]}}$$

$$= \frac{-10 + 3\cdot 0 - 8\cdot 0}{\sqrt{25 + 0 + 2\cdot 0}} = \frac{-10}{\sqrt{25}} = \frac{-10}{5} = \boxed{-2}$$

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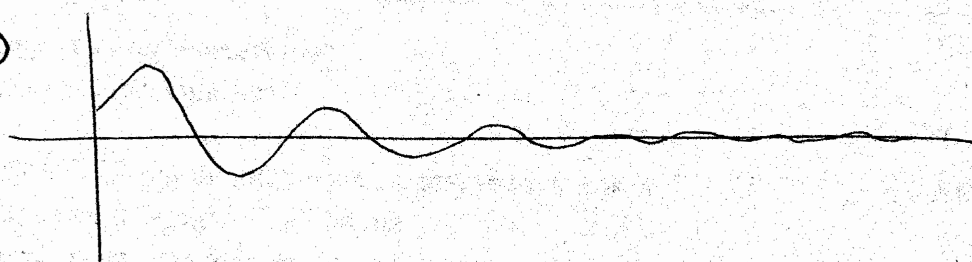
(2) c) horizontal asymptotes only

d) As $x \to \infty$, the horizontal asymptote is $y=2$.
As $x \to -\infty$, the horizontal asymptote is $y=-2$.

* You must show analytic work on quizzes or exams *

(3) $f(x) = \frac{\sin x}{\sqrt{x}}$

a)



b) $\lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} = \boxed{0}$ because y-coords approach 0 as $x \to \infty$.

c) Squeeze theorem:

$$-1 \le \sin x \le 1$$

$$\frac{-1}{\sqrt{x}} \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$$

$$\lim_{x\to\infty} \frac{-1}{\sqrt{x}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

$$\lim_{x\to\infty} \frac{-1}{\sqrt{x}} \le \lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} \le \lim_{x\to\infty} \frac{+1}{\sqrt{x}}$$

$$0 \le \lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} \le 0.$$

$$\therefore \lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} = 0.$$

d) horizontal asymptote $\boxed{y=0}$ (as $x \to \infty$)
no oblique asymptote

e) Graph is not real for $x<0$, so no asymptote or limit

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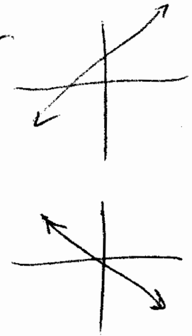
Recall from Precalculus

END BEHAVIOR

deg 1

$f(x) = ax + b$



if $a > 0$ $\begin{cases} \lim_{x \to \infty} f(x) = +\infty \\ \lim_{x \to -\infty} f(x) = -\infty \end{cases}$

if $a < 0$ $\begin{cases} \lim_{x \to +\infty} f(x) = -\infty \\ \lim_{x \to -\infty} f(x) = +\infty \end{cases}$

deg 3

$f(x) = ax^3 + bx^2 + cx + d$

$a > 0$

$a < 0$

same limits as degree 1

deg odd

$f(x) = ax^n + \dots$

n odd

same limits as deg 1

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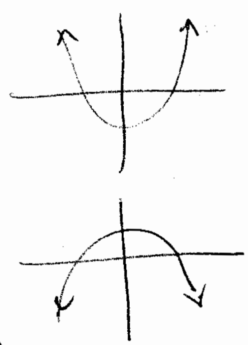
deg 2

$f(x) = ax^2 + bx + c$

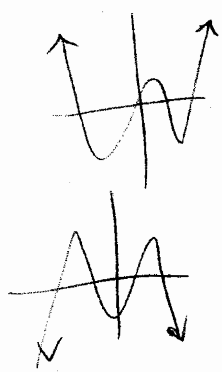


if $a > 0$ $\quad \lim_{x \to \infty} f(x) = +\infty$, $\lim_{x \to -\infty} f(x) = +\infty$

if $a < 0$ $\quad \lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = -\infty$

deg 4

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$



$a > 0$

$a < 0$

deg even

$f(x) = ax^n + \dots$

n even

Same limits as deg 2

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(4) $f(x) = 3x^4 - 6x^2 + x - 10$

a) $\lim_{x \to \infty} f(x)$

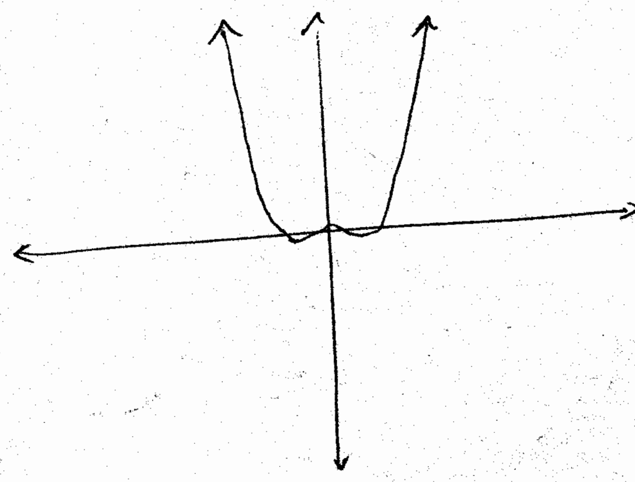
$= \lim_{x \to \infty} 3x^4$

$= \boxed{+\infty}$ DNE unbounded

leading term

$n = 4$ even

$a_n = 3$ positive



b) $\lim_{x \to -\infty} f(x)$

$= \lim_{x \to -\infty} 3x^4$

$= \boxed{+\infty}$ DNE unbounded

(5) $f(x) = -3x^5 - 6x^2 + x - 10$

a) $\lim_{x \to \infty} f(x)$

$= \lim_{x \to \infty} -3x^5$

$= \boxed{-\infty}$ DNE unbounded

leading term

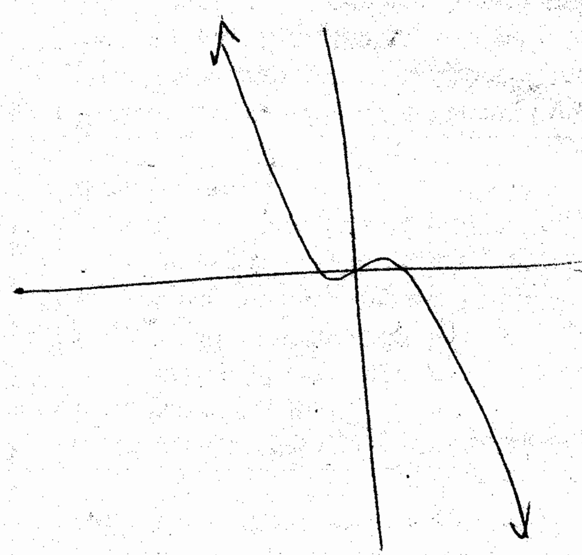
$n = 5$

$a_n = -3$ neg

b) $\lim_{x \to -\infty} f(x)$

$= \lim_{x \to -\infty} -3x^5$

$= \boxed{+\infty}$ DNE unbounded



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⑥ $f(x) = \dfrac{3x-4}{x^2+3}$

a) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\frac{3x}{x^2} - \frac{4}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}}$

$= \lim_{x\to\infty} \dfrac{\frac{3}{x} - \frac{4}{x^2}}{1 + \frac{3}{x^2}}$

$= \dfrac{0-0}{1+0}$

$= \boxed{0}$ deg denom > deg num

b) $\lim_{x\to-\infty} f(x) = \boxed{0}$ same as $\to +\infty$.

c) horizontal asymptote $\boxed{y=0}$

d) no oblique

⑦ $f(x) = \dfrac{3x^3-4}{2x+3}$

a) $\lim_{x\to\infty} \dfrac{\frac{3x^3}{x} - \frac{4}{x}}{\frac{2x}{x} + \frac{3}{x}}$

always divide by power of leading term in denominator.

$= \lim_{x\to\infty} \dfrac{3x^2 - \frac{4}{x}}{2 + \frac{3}{x}}$

$= \lim_{x\to\infty} \dfrac{3x^2}{2}$

leading term
$n=2$
$a_n=3$ positive

$= \boxed{+\infty,\ \text{DNE unbounded}}$

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b) $\lim_{x \to -\infty} f(x)$

$= \lim_{x \to -\infty} 3x^2$

$= \boxed{+\infty, \text{ DNE, unbounded}}$

c) oblique asymptote

d)

$$\begin{array}{r} \frac{3}{2}x^2 - \frac{9}{4}x + \frac{27}{8} \\ 2x+3 \,\overline{)\, 3x^3 + 0x^2 + 0x - 4} \\ 3x^3 + \frac{9}{2}x^2 \\ \hline -\frac{9}{2}x^2 + 0x \\ -\frac{9}{2}x^2 - \frac{27}{4}x \\ \hline \frac{27}{4}x - 4 \\ \frac{27}{4}x + \frac{81}{8} \\ \hline -\frac{113}{8} \end{array}$$

$\frac{3x^3}{2x} = \frac{3}{2}x^2$

$\frac{-\frac{9}{2}x^2}{2x} = -\frac{9}{2} \cdot \frac{1}{2}x = -\frac{9}{4}x$

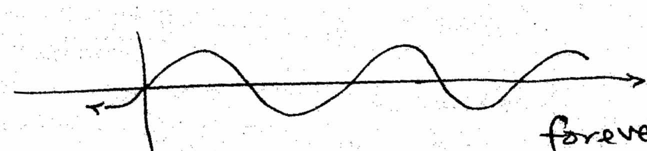
$\frac{\frac{27}{4}x}{2x} = \frac{27}{8}$

oblique asymptote

$y = \frac{3}{2}x^2 - \frac{9}{4}x + \frac{27}{8}$

(8) $f(x) = \sin x$

a) $\lim_{x \to \infty} \sin x = \boxed{\text{DNE oscillates}}$



forever →

b) $\lim_{x \to -\infty} \sin x = \boxed{\text{DNE oscillates}}$

c) neither oblique nor horizontal

d) N/A.